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Fatma Ghribi, Frédéric Klopp

LOCALIZATION FOR THE RANDOM DISPLACEMENT MODEL AT WEAK DISORDER

FATMA GHRIBI AND FRÉDÉRIC KLOPP

Abstract. This paper is devoted to the study of the random displacement model on $\mathbb{R}^d$. We prove that, in the weak displacement regime, Anderson and dynamical localization hold near the bottom of the spectrum under a generic assumption on the single site potential and a fairly general assumption on the support of the possible displacements. This result follows from the proof of the existence of Lifshitz tail and of a Wegner estimate for the model under scrutiny.

Résumé. Cet article est consacré à l'étude d'un modèle de petits déplacements aléatoires. Sous une hypothèse générique sur le potentiel de simple site et des hypothèses assez générales sur les déplacements autorisés, on démontre que le bas du spectre est exponentiellement et dynamiquement localisé dans la limite des petits déplacements. La preuve repose sur la preuve d'une estimée de Lifshitz et d'une estimée de Wegner pour le modèle étudié.

0. Introduction

We consider the following random displacement model

$$H_{\lambda,\omega} = -\Delta + p + q_{\lambda,\omega} \text{ where } q_{\lambda,\omega}(x) = \sum_{\gamma\in\mathbb{Z}^d} q(x-\gamma-\lambda\omega_\gamma), \tag{0.1}$$

acting on $L^2(\mathbb{R}^d)$. We assume the following:

- **(H.0.0):** The potential p is a real valued, $\mathbb{Z}^d$-periodic function.
- **(H.0.1):** The single site potential q is a twice continuously differentiable, real valued function and compactly supported.
- **(H.0.2):** $\omega := (\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ is a collection of non trivial, independent, identically distributed, bounded random variables; let $K \subset \mathbb{R}^d$ be the support of their common distribution.
- **(H.0.3):** λ is a small positive coupling constant.

Under these assumptions, $H_{\lambda,\omega}$ is ergodic and, for all ω, $H_{\lambda,\omega}$ is self-adjoint on the standard Sobolev space $\mathcal{H}^2(\mathbb{R}^d)$. The theory of ergodic operators teaches us that the spectrum of $H_{\lambda,\omega}$ is ω-almost surely independent of ω (see e.g. [11, 19]); we denote it by Σ_λ .
Our assumptions on $q_{\lambda,\omega}$ imply that Σ_λ is bounded below. Define $E_\lambda := \inf \Sigma_\lambda$.

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The goal of the present paper is to study the nature of the spectrum of $H_{\lambda,\omega}$ near E_λ. A result typical of the class of results that we will prove is

Theorem 0.1. *Assume p is not constant and that the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are uniformly distributed in the unit ball in $\mathbb{R}^d$.*
Then, there exists $\varepsilon_0 > 0$ and $\lambda_0 > 0$ such that, for a generic single site potential q such that $\|q\|_\infty \leq \varepsilon_0$, for $\lambda \in (0, \lambda_0]$, Anderson and strong dynamical localization near the bottom of the spectrum E_λ. Namely, there exist $E_{\lambda,1} > E_\lambda$ such that $H_{\lambda,\omega}$ has dense pure point spectrum on $[E_\lambda, E_{\lambda,1}]$ almost surely, and each eigenfunction associated to an energy in this interval decays exponentially as $|x| \to \infty$, and strong dynamical localization holds in the same region.

For details on strong dynamical localization, we refer to[7].
When studying its spectral properties, an important feature of $H_{\lambda,\omega}$ is that it depends non monotonically (see e.g. [17]) on the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$, even if q is assumed to be sign-definite. As each of the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ is multidimensional, there cannot be a real monotonicity. Nevertheless, we exhibit a set of assumptions on the single site potential q and on the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ that guarantee that, for sufficiently small disorder λ,

- there exists a neighborhood of E_λ where $H_{\lambda,\omega}$ admits a Wegner estimate,
- $H_{\lambda,\omega}$ exhibits a Lifshitz tails at E_λ.

It is well known that such results then entail Anderson and dynamical localization near E_λ (see e.g. [7]).
Our assumptions are presumably not optimal; we show that they hold for a small generic q. We need to assume some regularity for the distribution of the random variables. As they are multi-dimensional, absolute continuity with respect to the d-dimensional Lebesgue measure is not necessary; actually, they can be concentrated on subsets of dimension one (see section 1.2.3). As for the support of the single site random variable, they can have a wide variety of shapes but need to satisfy a type of strict convexity condition at certain points; we refer to section 1.3 for more details.

Due to the non monotonicity of $H_{\lambda,\omega}$, few rigorous results are known for the random displacement model in dimension larger than 1.
For the one-dimensional displacement model, localization at all energies was proven in [2] and, with different methods and, under more general assumptions, in [6]. These proofs establish the Wegner estimate using two-parameter spectral averaging and use lower bounds on the Lyapunov exponent to replace the Lifshitz tails behavior.
For the multi-dimensional random displacement model, the only available result on localization prior to the present paper was [14] establishing the existence of a localized region for the semi-classical operator

$-h^2\Delta + p + q_{\lambda,\omega}$ when h is sufficiently small. The Wegner estimate was established through a careful analysis of quantum tunneling. The Lifshitz tails behavior was neither proved nor used in the energy region under consideration, because of the semi-classical regime, the model is in a large disorder regime.
It has been discovered recently that, for random displacement models, Lifshitz tails need not hold (see [4, 16]).
Related to the study of the occurrence of the Lifshitz tails, an important point is the study of the infimum of the almost sure spectrum and, in particular of the finite volume configurations of the random parameter, if any, that give rise to the same ground state energy. Such a study for non monotonous models has been undertaken recently in [3, 17]. In the present paper, we give an analysis of those configuration in the small displacement case.

1. The main results

For $n \geq 0$, let $\Lambda_n = [-n - 1/2, n + 1/2]^d$. For $(\omega_\gamma)_{\gamma\in\mathbb{Z}}$, define the differential expression

$$H_{\lambda,\omega,n} = -\Delta + p + \sum_{\beta\in(2n+1)\mathbb{Z}^d} \sum_{\gamma\in\mathbb{Z}^d/(2n+1)\mathbb{Z}^d} q(x - \beta - \gamma - \lambda\omega_\gamma). \tag{1.1}$$

Let $H^P_{\lambda,\omega,n}$ be restriction of $H_{\lambda,\omega,n}$ to the cube Λ_n with periodic boundary conditions. $H^P_{\lambda,\omega,n}$ has only discrete spectrum and is bounded from below. For $E \in \mathbb{R}$, the *integrated density of states* is, as usual, defined by

$$N_\lambda(E) = \lim_{n\to+\infty} \frac{1}{(2n+1)^d} \#\{\text{eigenvalues of } H^P_{\lambda,\omega,n} \text{ in } (-\infty, E]\}.$$

We refer to [11, 19] for details on this function and the proofs of various standard results.

1.1. **The assumptions.**

We now state our assumptions on the random potential. Therefore, we introduce the periodic operator obtained by shifting all the single site potentials by exactly the same amount i.e. for $\zeta \in K$ (see assumption (H.0.2)), let

$$H_\zeta = H_{\lambda,\overline{\zeta}} = -\Delta + p + \sum_{\gamma\in\mathbb{Z}^d} q(x - \gamma - \lambda\zeta). \tag{1.2}$$

Here and in the sequel, $\overline{\zeta}$ denotes the constant vector with entries all equal to ζ i.e. $\overline{\zeta} = (\zeta)_{\gamma\in\mathbb{Z}^d}$.
The spectrum of the $\mathbb{Z}^d$-periodic operator H_ζ is purely absolutely continuous; it is a union of intervals (see e.g. [20]). Let $E(\lambda, \zeta)$ be the infimum of this spectrum. As $E(\lambda, \zeta)$ is the bottom of the spectrum of the periodic operator H_ζ, we know that it is a simple Floquet eigenvalue associated to the Floquet quasi-momentum $\theta = 0$ (see section 2.1

for more details); hence, it is a twice continuously differentiable function of ζ.
We assume that

(H.1.1): there exits $\lambda_0 > 0$ such that, for $\lambda \in (0, \lambda_0)$, there exists a unique point $\zeta(\lambda) \in K$ so that

$$E(\lambda, \zeta(\lambda)) = \min_{\zeta \in K} E(\lambda, \zeta);$$

(H.1.2): there exists $\alpha_0 > 0$ such that, for $\lambda \in (0, \lambda_0)$ and $\zeta \in K$, one has

$$\nabla_\zeta E(\lambda, \zeta(\lambda)) \cdot (\zeta - \zeta(\lambda)) \geq \alpha_0 \, \lambda \, |\zeta - \zeta(\lambda)|^2. \tag{1.3}$$

In section 1.3, we discuss concrete conditions on p, q and K that ensure that assumption (H.1) is valid. We now turn to our main results.

1.2. **The results.** We start with a description of the realizations of the random potential where the infimum of the almost sure spectrum is attained. Then, we state our results on Lifshitz tails, a Wegner estimate and the result on localization.

1.2.1. *The infimum of the almost sure spectrum.* Of course, as Σ_λ is the almost sure spectrum, almost all realizations have their infimum as the infimum of the spectrum. The realizations we are interested in are those that attain this infimum when restricted to a finite volume. In the present paper, we construct these restrictions using periodic boundary conditions, actually considering periodic realizations of the random potential. In [3, 4, 17, 16], the restrictions were performed using Neumann boundary conditions.
We define periodic configurations of the random potential. Fix $n \geq 0$ and, for $(\omega_\gamma)_{\gamma \in \mathbb{Z}^d/(2n+1)\mathbb{Z}^d}$, consider the differential operator $H_{\lambda,\omega,n}$ defined by (1.1) with domain $\mathcal{H}^2(\mathbb{R}^d)$. It is $(2n+1)\mathbb{Z}^d$-periodic; let $E_0^n(\lambda\omega)$ be its ground state energy i.e. the infimum of its spectrum.
One has

Theorem 1.1. *Under assumptions (H.0) and (H.1), there exists $\lambda_0 > 0$ such that, for any $n \geq 0$, for $\lambda \in (0, \lambda_0]$, on $K^{(2n+1)^d}$, the function $\omega \mapsto E_0^n(\lambda\omega)$ reaches its infimum $E(\lambda, \zeta(\lambda))$ at a single point, the point $\omega = (\zeta(\lambda))_{\gamma \in \mathbb{Z}^d/(2n+1)\mathbb{Z}^d}$.*

So, when it comes to finding the "ground state" of our random system, for small λ, the Hamiltonian behaves as if it were monotonous in the random variables $(\omega_\gamma)_\gamma$.
By the standard characterization of the almost sure spectrum in terms of the spectra of the periodic approximations (see e.g. [19]), for λ sufficiently small, one has that $E_\lambda = \inf \Sigma_\lambda = E(\lambda, \zeta(\lambda))$.

1.2.2. *The Lifshitz tails.* As a consequence of the determination of the minimum, we obtain

Theorem 1.2. *Under assumptions (H.0) and (H.1), there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$,*

$$\lim_{E \to E_\lambda} \frac{\log|\log(N_\lambda(E) - N_\lambda(E_\lambda)|}{\log(E - E_\lambda)} \leq -\frac{d}{2}$$

Moreover, if the common distribution of the random variables $(\omega_\gamma)_\gamma$ is such that, for all λ, ε and δ positive sufficiently small, one has

$$\mathbb{P}(\{|\omega_0 - \zeta(\lambda)| \leq \varepsilon\}) \geq e^{-\varepsilon^{-\delta}},$$

then

$$\lim_{E \to E_\lambda} \frac{\log|\log(N_\lambda(E) - N_\lambda(E_\lambda)|}{\log(E - E_\lambda)} = -\frac{d}{2}$$

The Lifshitz tail behavior is well known for monotonous alloy type models. It has also been discovered recently that, for general displacement or non monotonous alloy type models, this behavior need not hold (see [4, 17, 16]).

1.2.3. *The Wegner estimate.* A Wegner estimate is an estimate on the probability that a restriction of the random Hamiltonian to a cube admits an eigenvalue in a fixed energy interval. Clearly, the estimate should grow with the size of the cube and decrease with the length of the interval in which one looks for eigenvalues.
The restrictions we choose are the periodic ones i.e those defined at the beginning of section 1.2. We assume that

(H.2): There exists $C > 0$ such that, for λ sufficiently small, one has $E_\lambda \leq E_0 - \lambda/C$.

Clearly, Theorem 1.1 shows that this assumption is a consequence of assumptions (H.0) and (H.1).
For the alloy type models, it is well known that a Wegner estimate will hold only under a regularity assumption. We now turn to the corresponding assumption for our displacement model. We keep the notations of section 1.2.1. Consider the polar decomposition of the random variable ω_0, say $\omega_0 = r(\omega_0)\sigma(\omega_0)$. For $\sigma \in \mathbb{S}^{d-1}$, define $r_\sigma(\omega_0)$, the random variable $r(\omega_0)$ conditioned on $\sigma(\omega_0) = \sigma$.
We assume that

(H.3): for almost all $\sigma \in \mathbb{S}^{d-1}$, the distribution of $r_\sigma(\omega_0)$ admits a density with respect to the Lebesgue measure, say, h_σ that itself is absolutely continuous with respect to the Lebesgue measure; moreover, one has

$$\text{ess-sup}_{\sigma \in \mathbb{S}^{d-1}} \|h'_\sigma\|_\infty < +\infty. \tag{1.4}$$

Remark 1.1. Assumption (H.3) will hold for example if

- the random variable admit a density that is continuously differentiable on its support;
- the random variable is supported on a submanifold of dimension $1 \leq d' \leq d$, and on this submanifold, it admits continuously differentiable density.

We prove

Theorem 1.3. *Under assumptions (H.0), (H.2) and (H.3), for any* $\nu \in (0,1)$*, there exists* $\lambda_0 > 0$ *such that, for* $\lambda \in (0, \lambda_0]$*, there exists* $C_\lambda > 0$ *such that, for all* $E \in [E_\lambda, E_\lambda + \lambda/C]$ *and* $\varepsilon > 0$ *such that*

$$\mathbb{P}(dist(\sigma(H^P_{\lambda,\omega,n}), E) \leq \varepsilon) \leq C_\lambda \varepsilon^\nu n^d. \tag{1.5}$$

The result is essentially a quite simple consequence of Theorem 6.1 of [9]; the modifications are indicated in section 2.5.
In the case of monotonous random operators, under our smoothness assumptions for the distribution of the random variables, the estimate (1.5) can be improved in the sense that the power ν can be taken equal to 1 (see [5]). It seems reasonable to think that the same holds true for most non monotonous models; to our knowledge, no proof of this fact exists.
A Wegner estimate of the type (1.5) implies a minimal regularity for N_λ, the integrated density of states of $H_{\lambda,\omega}$ in the low energy region. Indeed, one proves

Corollary 1.1. *Under the assumptions of Theorem 1.3, for any* $\nu \in (0,1)$*, the integrated density of states* N_λ *is* ν*-Hölder continuous is the region* $[E_\lambda, E_\lambda + \lambda/C]$ *defined in Theorem 1.3.*

1.2.4. *Localization.* Once Theorems 1.2 and 1.3 are proved, localization follows by the now standard multiscale argument (see e.g. [7])

Theorem 1.4. *Under assumptions (H.0), (H.1) and (H.3), there exists* $\lambda_0 > 0$ *such that, for* $\lambda \in (0, \lambda_0]$*, Anderson and strong dynamical localization near the bottom of the spectrum. Namely, there exist* $E_{\lambda,1} > E_\lambda$ *such that* $H_{\lambda,\omega}$ *has dense pure point spectrum on* $[E_\lambda, E_{\lambda,1}]$ *almost surely, and each eigenfunction associated to an energy in this interval decays exponentially as* $|x| \to \infty$*, and strong dynamical localization holds in the same region.*

We omit the details of the proofs of this result. We only note that the Combes-Thomas estimate and the decomposition of resolvents in the multiscale argument work for the random displacement model in the same way as for alloy type models.

1.3. **The validity of assumption (H.1).** Let us now describe some concrete conditions on q and K that ensure that assumption (H.1) does hold. Let $H_0 = H_{\lambda,0}$ be defined by (1.2) for $\zeta = 0$. The spectrum of this operator is purely absolutely continuous; it is a union of intervals (see e.g. [20]). Let E_0 be the infimum of this spectrum and φ_0 be the solution to the following spectral problem

$$\begin{cases} H_0\varphi_0 = E_0\varphi_0, \\ \forall \gamma \in \mathbb{Z}^d, \ \varphi_0(x+\gamma) = \varphi_0(x). \end{cases} \tag{1.6}$$

This solution is unique up to a constant; it can be chosen positive and normalized (see [12, 21]). We will then call it the ground state for H_0. Recall that K is the essential support of the random variables $(\omega_\gamma)_\gamma$; thus $K \subset \mathbb{R}^d$.
We prove

Proposition 1.1. *Assume that K is*

- *either a convex set with C^2-boundary such that all its principal curvatures are positive at all points,*
- *or the boundary of such a convex set,*

and that

$$v(q) := -\int_{\mathbb{R}^d} \nabla q(x)|\varphi_0(x)|^2 dx \neq 0, \tag{1.7}$$

Then, assumption (H.1) holds.

For a fixed periodic potential p that is not constant, by perturbation theory, it is not difficult to see that condition (1.7) is satisfied for a generic small q. Indeed, if ψ_0 is the ground state for $-\Delta + p$ (in the sense defined above), as ψ_0 is positive, its modulus is constant if and only if it is constant. In which case, the eigenvalue equation (1.6) tells us that p is constant, identically equal to E_0. So we may assume that ψ_0 is not constant, one can then find q smooth and compactly supported such that (1.7) holds. Indeed, by integration by parts,

$$w(q) := \int_{\mathbb{R}^d} \partial_i q(x)\psi_0^2(x)dx = 2\int_{\mathbb{R}^d} q(x)\psi_0(x)\partial_i\psi_0(x)dx$$

which vanishes for all smooth compactly supported functions if and only if $\partial_i\psi_0$ vanishes identically. Hence, $w(q)$ vanishes for all q small, smooth and compactly supported if and only if ψ_0 is a constant (as $q \mapsto w(q)$ is linear).
As φ_0 is the ground state for the operator $-\Delta+p+\sum_\gamma q(\cdot-\gamma)$ and this ground state is a real analytic function of the potential q, the difference $\psi_0 - \varphi_0$ is small for q small. So, if we pick q_0 such that $w(q_0) \neq 0$, for ε small and $q = \varepsilon q_0$, we know that $v(q)$ does not vanish i.e. (1.7) is satisfied.

By Proposition 1.1 and Remark 1.1, it is clear now that Theorem 0.1 is a consequence of Theorem 1.4.

Let us now give another assumption on K under which (H.1) holds. We prove

Proposition 1.2. *Assume that (1.7) is satisfied and that the set K satisfies that, there exists $\varepsilon > 0$ and $\zeta_0 \in K$, such that, for all $\zeta \in K$ and $|v - v(q)| < \varepsilon$, one has*

$$v \cdot (\zeta - \zeta_0) \geq 0.$$

Then, assumption (H.1) holds. Moreover, for λ small, the minimum $\zeta(\lambda)$ satisfies $\zeta(\lambda) = \zeta_0$.

Before we proceed to the proofs of Propositions 1.1 and 1.2, let us compare our setting to the one studied in [3, 4, 16]. In those studies, assumption (1.7) but also assumption (H.1) are not fulfilled. Indeed, there, p and q are assumed to be reflection symmetric with respect to the coordinate planes i.e. for any $\sigma = (\sigma_1, \ldots, \sigma_d) \in \{0,1\}^d$ and any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$q(x_1, \ldots, x_d) = q((-1)^{\sigma_1} x_1, \ldots, (-1)^{\sigma_d} x_d).$$

Hence, the potential $p(\cdot) + \sum_{\gamma} q(\cdot - \gamma)$ and the ground state φ_0 satisfy the same reflection symmetry. This implies that

$$\int_{\mathbb{R}^d} \nabla q(x) |\varphi_0(x)|^2 dx = - \int_{\mathbb{R}^d} \nabla q(x) |\varphi_0(x)|^2 dx = 0.$$

The fact that, in the setting of [3, 4, 16], assumption (H.1.1) is not satisfied is seen directly from those papers as the ground state of the periodic operator H_ζ reaches its minimum at 2^d values as soon as K is reflection symmetric.

1.4. **The proofs of Propositions 1.1 and 1.2.** Consider the mapping $\zeta \mapsto F(\lambda, \zeta) = \lambda^{-1} E(\lambda, \zeta)$ on some large ball B containing K. As $E(\lambda, \zeta)$ is a simple Floquet eigenvalue associated to the normalized Floquet eigenvector $\varphi_0(\lambda, \zeta, 0)$ (see section 2.1), we can compute the gradient of F in the ζ-variable using the Feynman-Hellmann formula to obtain

$$\nabla_\zeta F(\lambda, \zeta) = - \int_{\mathbb{R}^d} \nabla q(x - \lambda\zeta) |\varphi_0(\lambda, \zeta, 0; x)|^2 dx.$$

Hence,

$$\sup_{\zeta \in B} |\nabla_\zeta F(\lambda, \zeta) - v(q)| \underset{\lambda \to 0}{\to} 0 \tag{1.8}$$

Proof of Proposition 1.1. Assume first that K is a convex set satisfying the assumptions of Proposition 1.1. Using the rectification theorem (see e.g. [1]), assumption (1.7) and equation (1.8) guarantee that, for λ small, one can find a C^2-diffeomorphism, say Ψ_λ, from B to $\Psi_\lambda(B)$ such that $|\Psi_\lambda - \mathrm{Id}|_{C^2} \to 0$ when $\lambda \to 0$ and

$$\nabla_\zeta (F(\lambda, \Psi_\lambda(\zeta))) = v(q).$$

Now, assume that K is a convex set with a C^2-boundary having all its principal curvatures positive at all points. Then, for λ small, the set $K_\lambda = \Psi_\lambda^{-1}(K)$ also is convex with a C^2-boundary having all its principal curvatures positive at all points; moreover, the curvatures are bounded away from 0 independently of λ for λ small.

On the convex set K_λ, the affine function $G(\zeta) := F(\lambda, \Psi_\lambda(\zeta)) = v(q) \cdot \zeta + C_\lambda$ reaches its infimum at a single point, say $\tilde{\zeta}(\lambda) = \Psi_\lambda^{-1}(\zeta(\lambda))$, $\zeta(\lambda) \in \partial K$.

Hence, we have that, for $\zeta \in K \setminus \{\zeta(\lambda)\}$, $F(\lambda, \zeta) > F(\lambda, \zeta(\lambda))$. The convexity of K ensures that, for $\zeta \in K \setminus \{\zeta(\lambda)\}$, one has

$$\nabla_\zeta F(\lambda, \zeta(\lambda)) \cdot (\zeta - \zeta(\lambda)) \geq 0. \tag{1.9}$$

Indeed, as K is convex, for $\nu \in (0,1)$ and $\zeta \in K \setminus \{\zeta(\lambda)\}$, one has $\zeta_\nu = \nu\zeta + (1-\nu)\zeta(\lambda) \in K \setminus \{\zeta(\lambda)\}$; thus $F(\lambda, \zeta_\nu) > F(\lambda, \zeta(\lambda))$. Taking the right hand side derivative of $\nu \mapsto F(\lambda, \zeta_\nu)$ at $\nu = 0$ yields (1.9).

The strict convexity of K, guaranteed by the positivity of the principal curvatures of ∂K, ensures that, for $\zeta \in K \setminus \{\zeta(\lambda)\}$, one has

$$\nabla_\zeta F(\lambda, \zeta(\lambda)) \cdot (\zeta - \zeta(\lambda)) > 0. \tag{1.10}$$

Indeed, assume that for some $\zeta_0 \in K \setminus \{\zeta(\lambda)\}$, (1.10) is not satisfied i.e $\nabla_\zeta F(\lambda, \zeta(\lambda)) \cdot (\zeta_0 - \zeta(\lambda)) = 0$. As K is strictly convex, K contains a cone of the form $\{\zeta(\lambda) + r(\zeta_0 - \zeta(\lambda)) + rw;\ \|w\| \leq 1,\ r \in [0, r_0]\}$ for some small $r_0 > 0$. Picking w such that $\nabla_\zeta F(\lambda, \zeta(\lambda)) \cdot w < 0$, one constructs $\zeta' \in K$ such that $\nabla_\zeta F(\lambda, \zeta(\lambda)) \cdot (\zeta' - \zeta(\lambda)) < 0$ which contradicts (1.9).

To show (1.3), it suffices to show that, for $\zeta \in K$,

$$\nabla_\zeta F(\lambda, \zeta(\lambda)) \cdot (\zeta - \zeta(\lambda)) \geq \frac{1}{C_0} |\zeta - \zeta(\lambda)|^2. \tag{1.11}$$

Let H_λ be the hyperplane orthogonal to $\nabla_\zeta F(\lambda, \zeta(\lambda))$ at $\zeta(\lambda)$. It intersects K at $\zeta(\lambda)$ and K is contained in one of the half-spaces defined by this hyperplane. Thus, the hyperplane is tangent to K at $\zeta(\lambda)$ (see e.g. [10]). Hence, there exists $\alpha_0 > 0$ such that, for $\zeta \in K$, one has

$$\nabla_\zeta F(\lambda, \zeta(\lambda)) \cdot (\zeta - \zeta(\lambda)) \geq \alpha_0\, d(\zeta, H_\lambda)^2 \tag{1.12}$$

where $d(\zeta, H_\lambda)$ denotes the distance from ζ to H_λ. The constant α_0 can be chosen independent of λ for λ small as the principal curvatures of ∂K are uniformly positive. Now, if $u = \|\nabla_\zeta F(\lambda, \zeta(\lambda))\|^{-1} \nabla_\zeta F(\lambda, \zeta(\lambda))$, for $\zeta \in K$ as K is compact, one has

$$\begin{aligned} \nabla_\zeta F(\lambda, \zeta(\lambda)) \cdot (\zeta - \zeta(\lambda)) &= \|\nabla_\zeta F(\lambda, \zeta(\lambda))\| \, [u \cdot (\zeta - \zeta(\lambda))] \\ &\geq \alpha_0 \, [u \cdot (\zeta - \zeta(\lambda))]^2. \end{aligned} \tag{1.13}$$

As $|\zeta - \zeta(\lambda)|^2 = d(\zeta, H_\lambda)^2 + [u \cdot (\zeta - \zeta(\lambda))]^2$, the lower bounds (1.12) and (1.13) imply (1.11).

To deal with the case when K is the boundary of a convex set, we only need to do the analysis done above for the convex hull of K and notice that the minimum is attained on K the boundary of this convex hull. This completes the proof of Proposition 1.1. □

Proof of Proposition 1.2. By assumption, for $\zeta \in K$ and $|v - v(q)| < \varepsilon$, one has $v \cdot (\zeta - \zeta_0) \geq 0$. Hence, as K is compact, there exists $c > 0$ such that, for all $\zeta \in K$ and $|v - v(q)| < \varepsilon/2$, one has

$$v \cdot (\zeta - \zeta_0) \geq c|\zeta - \zeta_0|. \tag{1.14}$$

Let B be a closed ball centered in ζ_0 such that $K \subset B$. By (1.8), (1.14) implies that, for λ sufficiently small, for all $\tilde{\zeta} \in B$ and $\zeta \in K$, one has

$$\nabla_\zeta F(\lambda, \tilde{\zeta}) \cdot (\zeta - \zeta_0) \geq c|\zeta - \zeta_0|.$$

Hence,

$$\begin{aligned} F(\lambda, \zeta) - F(\lambda, \zeta_0) &= \int_0^1 \nabla_\zeta F(\lambda, \zeta_0 + t(\zeta - \zeta_0)) \cdot (\zeta - \zeta_0) dt \\ &\geq c|\zeta - \zeta_0|. \end{aligned}$$

So ζ_0 is the unique minimum of $\zeta \mapsto F(\lambda, \zeta)$ in K i.e. for λ sufficiently small, $\zeta(\lambda) = \zeta_0$. Using again the boundedness of K, we get the estimate (1.3) of assumption (H.1). This completes the proof of Proposition 1.2. □

2. The reduction to a discrete model

In this section, we prove the results announced in section 1.2.1. Therefore, we will use the Floquet decomposition for periodic operators to reduce our operator to some discrete model in the way it was done in [15, 8].

2.1. **Floquet theory.** Pick $\zeta \in K$ and let H_ζ be the $\mathbb{Z}^d$-periodic operator defined by (1.2). For $\theta \in \mathbb{T}^* := \mathbb{R}^d/(2\pi\mathbb{Z}^d)$ and $u \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz space of rapidly decaying functions, following [20], we define

$$(Uu)(\theta, x) = \sum_{\gamma \in \mathbb{Z}^d} e^{i\gamma\cdot\theta} u(x - \gamma)$$

which can be extended as a unitary isometry from $L^2(\mathbb{R}^d)$ to $\mathcal{H} := L^2(K_0 \times \mathbb{T}^*)$ where $K_0 = (-1/2, 1/2]^d$ is the fundamental cell of $\mathbb{Z}^d$. The inverse of U is given by

$$\text{for } v \in \mathcal{H},\ (U^*v)(x) = \frac{1}{\mathrm{Vol}(\mathbb{T}^*)} \int_{\mathbb{T}^*} v(\theta, x) d\theta.$$

As $H_{\lambda,\bar{\zeta}}$ is $\mathbb{Z}^d$-periodic, $H_{\lambda,\bar{\zeta}}$ admits the Floquet decomposition

$$U H_{\lambda,\bar{\zeta}} U^* = \int_{\mathbb{T}^*}^{\oplus} H_{\lambda,\bar{\zeta}}(\theta) d\theta$$

where $H_{\lambda,\bar{\zeta}}(\theta)$ is the differential operator $H_{\lambda,\bar{\zeta}}$ acting on $\mathcal{H}_\theta$ with domain $\mathcal{H}^2_\theta$ where

- for $v \in \mathbb{R}^d$, $\tau_v :\ L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ denotes the "translation by v" operator i.e for $\varphi \in L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, $(\tau_v \varphi)(x) = \varphi(x - v)$;

- $\mathcal{D}'_\theta$ is the space θ-quasi-periodic distribution in $\mathbb{R}^d$ i.e the space of distributions $u \in \mathcal{D}'(\mathbb{R}^d)$ such that, for any $\gamma \in \mathbb{Z}^d$, we have $\tau_\gamma u = e^{-i\gamma\cdot\theta}u$. Here $\theta \in \mathbb{T}^*$;
- $\mathcal{H}^k_{\mathrm{loc}}(\mathbb{R}^d)$ is the space of distributions that locally belong to $\mathcal{H}^k(\mathbb{R}^d)$ and we define $\mathcal{H}^k_\theta = \mathcal{H}^k_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathcal{D}'_\theta$;
- for $k = 0$, we define $\mathcal{H}_\theta = \mathcal{H}^0_\theta$ and identify it with $L^2(K_0)$; equipped with the L^2-norm over K_0, it is a Hilbert space; the scalar product will be denoted by $\langle\cdot,\cdot\rangle_\theta$.

We know that $H_{\lambda,\overline{\zeta}}(\theta)$ is self-adjoint and has a compact resolvent; hence its spectrum is discrete. Its eigenvalues repeated according to multiplicity, called Floquet eigenvalues of $H_{\lambda,\overline{\zeta}}$, are denoted by

$$E_0(\lambda,\zeta,\theta) \le E_1(\lambda,\zeta,\theta) \le \cdots \le E_n(\lambda,\zeta,\theta) \to +\infty.$$

The functions $((\lambda,\zeta,\theta) \mapsto E_n(\lambda,\zeta,\theta))_{n\in\mathbb{N}}$ are Lipschitz-continuous in the variable θ; they are even analytic in (λ,ζ,θ) when they are simple eigenvalues.
Define $\varphi_n(\lambda,\zeta,\theta)$ to be a normalized eigenvector associated to the eigenvalue $E_n(\lambda,\zeta,\theta)$. The family $(\varphi_n(\lambda,\zeta,\theta))_{n\ge0}$ is chosen so as to be a Hilbert basis of $\mathcal{H}_\theta$. If $E_n(\lambda_0,\zeta_0,\theta_0)$ is a simple eigenvalue, the function $(\lambda,\zeta,\theta) \mapsto \varphi_n(\lambda,\zeta,\theta)$ is analytic near $(\lambda_0,\zeta_0,\theta_0)$.
It is well known (see e.g. [12]) that, for given λ and ζ, the eigenvalue $E_0(\lambda,\zeta,\theta)$ reaches its minimum at $\theta = 0$, and that it is simple for θ small.

2.2. **The reduction procedure.** Recall that the $(\varphi_n(\lambda,\zeta,\theta))_{n\ge0}$ are the Floquet eigenvectors of $H_{\lambda,\overline{\zeta}}$. Let $\Pi_{\lambda,\zeta,0}(\theta)$ and $\Pi_{\lambda,\zeta,+}(\theta)$ respectively denote the orthogonal projections in $\mathcal{H}_\theta$ on the vector spaces respectively spanned by $\varphi_0(\lambda,\zeta,\theta)$ and $(\varphi_n(\lambda,\zeta,\theta))_{n\ge1}$. Obviously, these projectors are mutually orthogonal and their sum is the identity for any $\theta \in \mathbb{T}^*$.
Define $\Pi_{\lambda,\zeta,\alpha} = U^*\Pi_{\lambda,\zeta,\alpha}(\theta)U$ where $\alpha \in \{0,+\}$. $\Pi_{\lambda,\zeta,\alpha}$ is an orthogonal projector on $L^2(\mathbb{R}^d)$ and, for $\gamma \in \mathbb{Z}^d$, we have $\tau^*_\gamma\Pi_{\lambda,\zeta,\alpha}\tau_\gamma = \Pi_{\lambda,\zeta,\alpha}$. It is clear that that $\Pi_{\lambda,\zeta,0} + \Pi_{\lambda,\zeta,+} = Id_{L^2(\mathbb{R}^d)}$ and $\Pi_{\lambda,\zeta,0}$ and $\Pi_{\lambda,\zeta,+}$ are mutually orthogonal. For $\alpha \in \{0,+\}$, we set $\mathcal{E}_{\lambda,\zeta,\alpha} = \Pi_{\lambda,\zeta,\alpha}(L^2(\mathbb{R}^d))$. These spaces are invariant under translations by vectors in $\mathbb{Z}^d$ and $\mathcal{E}_{\lambda,\zeta,0}$ is of finite energy (see [15]).
For $u \in L^2(\mathbb{T}^*)$, we define

$$P_{\lambda,\zeta}(u) = U^*(u(\theta)\varphi_0(\lambda,\zeta,\theta)).$$

The mapping $P_{\lambda,\zeta} : L^2(\mathbb{T}^*) \to \mathcal{E}_{\lambda,\zeta,0}$ defines a unitary equivalence (see [15]); its inverse is given by

$$P^*_{\lambda,\zeta}(v) = \langle (Uv)(\theta), \varphi_0(\lambda,\zeta,\theta)\rangle, \quad v \in \mathcal{E}_{\lambda,0}.$$

One checks that $P_{\lambda,\zeta}P^*_{\lambda,\zeta} = \Pi_{\lambda,\zeta,0}$ and $P^*_{\lambda,\zeta}P_{\lambda,\zeta} = Id_{L^2(\mathbb{T}^*)}$.
The main result of this section is

Theorem 2.1. *Under assumptions (H.0) and (H.1), there exists $C_0 > 0$ such that, for any $\alpha > 0$, there exists $\lambda_0 > 0$ such that, for $\lambda \in (0, \lambda_0)$, for any $\zeta \in K$ and any $\omega = (\omega_\gamma)_{\gamma\in\mathbb{Z}^d} \in K^{\mathbb{Z}^d}$, one has*

$$\begin{aligned}\frac{1}{C_0}&\left(P_{\lambda,\zeta} h^-_{\lambda,\omega,\zeta} P^*_{\lambda,\zeta} + \Pi_{\lambda,\zeta,+}\right)\\ &\leq H_{\lambda,\omega} - E(\lambda,\zeta)\\ &\qquad\leq C_0\left(P_{\lambda,\zeta} h^+_{\lambda,\omega,\zeta} P^*_{\lambda,\zeta} + \tilde{H}_{\lambda,\overline{\zeta},+}\right)\end{aligned} \tag{2.1}$$

where

- $\tilde{H}_{\lambda,\overline{\zeta},+} = (H_{\lambda,\overline{\zeta}} - E(\lambda,\zeta))\Pi_{\lambda,\zeta,+}$.
- $h^\pm_{\lambda,\omega,\zeta}$ *is the random operator acting on* $L^2(\mathbb{T}^*)$ *defined by*

$$h^+_{\lambda,\omega,\zeta} = C_0\,\varpi(\cdot) + \lambda \sum_{\gamma\in\mathbb{Z}^d} \left[v(\lambda,\zeta)\cdot(\omega_\gamma - \zeta) + C_0\,\alpha\,\|\omega_\gamma - \zeta\|^2\right]\Pi_\gamma$$

$$h^-_{\lambda,\omega,\zeta} = \frac{1}{C_0}\,\varpi(\cdot) + \lambda \sum_{\gamma\in\mathbb{Z}^d} \left[v(\lambda,\zeta)\cdot(\omega_\gamma - \zeta) - C_0\,\alpha\,\|\omega_\gamma - \zeta\|^2\right]\Pi_\gamma$$

- ϖ *is the multiplication operator by the function*

$$\varpi(\theta) = \sum_{j=1}^d (1 - \cos(\theta_j)), \tag{2.2}$$

- Π_γ *is the orthogonal projector on* $e^{i\gamma\theta}$,
- *the vector* $v(\lambda,\zeta)$ *is given by*

$$v(\lambda,\zeta) = -\int_{\mathbb{R}^d} \nabla q(x - \lambda\zeta)|\varphi_0(\lambda,\zeta,0;x)|^2 dx = \frac{1}{\lambda}\nabla_\zeta E(\lambda,\zeta). \tag{2.3}$$

The proof of Theorem 2.1 is the content of section 3. We now use this result to derive Theorem 1.1 and 1.2.

2.3. **The characterization of the infimum of the almost sure spectrum.** We now prove Theorem 1.1. Using Theorem 2.1 for $\zeta = \zeta(\lambda)$, we see that, for λ sufficiently small, for $\omega \in K^{(2n+1)^d}$, one has

$$H_{\lambda,\omega,n} - E_\lambda \geq \frac{1}{C_0}\left(P_{\lambda,\zeta(\lambda)} h^-_{\lambda,\omega,\zeta(\lambda),\alpha,n} P^*_{\lambda,\zeta(\lambda)} + \Pi_{\lambda,\zeta(\lambda),+}\right). \tag{2.4}$$

Using (1.3) and (2.3), taking $C_0\alpha \leq \alpha_0/2$, we get

$$C_0\, h^-_{\lambda,\omega,\zeta(\lambda),n} \geq \varpi(\cdot) + \frac{C_0\lambda\,\alpha_0}{2} \sum_{\beta\in(2n+1)\mathbb{Z}^d}\ \sum_{\gamma\in\mathbb{Z}^d/(2n+1)\mathbb{Z}^d} \|\omega_\gamma - \zeta(\lambda)\|^2 \Pi_{\gamma+\beta}.$$

As the spectrum of $h^-_{\lambda,\omega,\zeta(\lambda),n}$ is non negative, the operator in the left hand side of (2.4) is clearly non negative; recall that $P_{\lambda,\zeta}P^*_{\lambda,\zeta} + \Pi_{\lambda,\zeta,+} = Id_{L^2}$, $\Pi_{\lambda,\zeta,+}$ is an orthogonal projector and $P^*_{\lambda,\zeta}$ is a partial unitary equivalence.

To prove Theorem 1.1, we will show that, if $\omega \neq (\zeta(\lambda))_{\gamma\in\mathbb{Z}^d/(2n+1)\mathbb{Z}^d}$,

then, there exists $c(\omega)>0$ such that $h^-_{\lambda,\omega\zeta(\lambda),n}\geq c(\omega)$. Therefore, recall that $h^-_{\lambda,\omega,\zeta(\lambda),n}$ is a periodic operator so we can do its Floquet decomposition in the same way as in section 2.1. In the present case, as we deal with a discrete model, the fiber operators will be finite dimensional matrices (see e.g. [13]); they can also be represented as the operator $h^-_{\lambda,\omega,\zeta(\lambda),n}$ acting on the finite dimensional space of linear combinations of the Dirac masses $(\delta_{2\pi k/(2n+1)+\theta})_{k\in\mathbb{Z}^d/(2n+1)\mathbb{Z}^d}$; the Floquet parameter θ belongs to $(2n+1)^{-1}\mathbb{T}^*$.
As $\varpi\geq 0$ and $h^-_{\lambda,\omega,\zeta(\lambda),n}-\varpi\geq 0$, 0 is in the spectrum of $h^-_{\lambda,\omega,\zeta(\lambda),n}$ if and only if there exists $\theta\in(2n+1)^{-1}\mathbb{T}^*$ and v, a linear combination of the Dirac masses $(\delta_{2\pi k/(2n+1)+\theta})_{k\in\mathbb{Z}^d/(2n+1)\mathbb{Z}^d}$ (seen as distributions on $\mathbb{T}^*$) such that $\varpi\cdot v=0$ and $h^-_{\lambda,\omega,\zeta(\lambda),n}v=0$. Now, $\varpi\cdot v=0$ implies that $\theta=0$ and $v=c\delta_0$. Hence, $h^-_{\lambda,\omega,\zeta(\lambda),n}v=0$ implies that

$$\sum_{\gamma\in\mathbb{Z}^d/(2n+1)\mathbb{Z}^d}\|\omega_\gamma-\zeta(\lambda)\|^2=0$$

i.e. $\omega=(\zeta(\lambda))_{\gamma\in\mathbb{Z}^d/(2n+1)\mathbb{Z}^d}$.
So we see that the function $\omega\mapsto E_0^n(\lambda\omega)$ reaches its infimum only at the point $\omega=(\zeta(\lambda))_{\gamma\in\mathbb{Z}^d/(2n+1)\mathbb{Z}^d}$. This completes the proof of Theorem 1.1. □

2.4. **The Lifshitz tails.** We now prove Theorem 1.2. Therefore, we again use the reduction given by Theorem 2.1.
Fix $\zeta=\zeta(\lambda)$. First, the operators $h^\pm_{\lambda,\omega,\zeta(\lambda)}$ are both standard discrete Anderson models and, as such, admit each integrated density of states that we denote by $N^\pm_r$. As we have seen in the previous section, their spectra are contained in $\mathbb{R}^+$.
The inequality (2.1) implies that, for λ sufficiently small and $E\in[0,1/C_0^2]$ where C_0 is the constant given in Theorem 2.1, one has

$$N^+_r(E/C_0)\leq N_\lambda(E_\lambda+E)\leq N^-_r(C_0\,E).$$

Now, Theorem 1.2 immediately follows from the existence of Lifshitz tail for the Anderson models $h^\pm_{\lambda,\omega,\zeta(\lambda)}$ (see e.g. [19]) which, in turn, follows from the facts that, for λ sufficiently small, under our assumptions, if $C_0\alpha\leq\alpha_0/2$, by (1.3), the random variables

$$\omega^\pm_\gamma=[v(\lambda,\zeta(\lambda))\cdot(\omega_\gamma-\zeta(\lambda))]\pm C_0\alpha\|\omega_\gamma-\zeta(\lambda)\|^2$$

are i.i.d, non negative, non trivial and 0 belongs to their support (see e.g. [19, 22]).
Now if the common distribution of the random variables $(\omega_\gamma)_\gamma$ is such that, for all λ, ε and δ positive sufficiently small, one has

$$\mathbb{P}(\{|\omega_0-\zeta(\lambda)|\leq\varepsilon\})\geq e^{-\varepsilon^{-\delta}},$$

then, by virtue of (1.3), for all λ, ε and δ positive sufficiently small, one has

$$\mathbb{P}(\{\omega_0^\pm \geq \varepsilon\}) \geq e^{-\varepsilon^{-\delta}}.$$

It is well known that, under this assumption, the Lifshitz exponent for the density of states of the discrete Anderson model is equal to $d/2$ (see e.g. [19]).
This completes the proof of Theorem 1.2. □

2.5. **The Wegner estimate.** We now prove Theorem 1.3 using the results of [9]. Let $H^P_{\lambda,r,\sigma,n}$ be the operator $H^P_{\lambda,\omega,n}$ where the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are written in polar coordinates i.e. $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d} = (r_\gamma(\omega)\,\sigma_\gamma(\omega))_{\gamma_\gamma\in\mathbb{Z}^d}$ where $r = (r_\gamma(\omega))_{\gamma\in\mathbb{Z}^d}$ has only non negative components and $\sigma = (\sigma_\gamma(\omega))_{\gamma\in\mathbb{Z}^d} \in [\mathbb{S}^{d-1}]^{\mathbb{Z}^d}$. Then, the basic observation is that

$$\mathbb{P}(\mathrm{dist}(\sigma(H^P_{\lambda,\omega,n}), E) \leq \varepsilon) = \mathbb{E}_\sigma\left(\mathbb{P}_r(\mathrm{dist}(\sigma(H^P_{\lambda,r,\sigma,n}), E) \leq \varepsilon |\, \sigma)\right) \tag{2.5}$$

where $\mathbb{P}_r(\cdot|\,\sigma)$ denotes the probability in the r-variable conditioned on σ, and $\mathbb{E}_\sigma$, the expectation in the σ-variable.
Now, fix $\sigma \in [\mathbb{S}^{d-1}]^{\mathbb{Z}^d}$. Using the notations of section 1.2.3, we write

$$H_{\lambda,\omega} = H_0 + \lambda \sum_{\gamma\in\mathbb{Z}^d} r_\sigma(\omega_\gamma) v_{\sigma_\gamma}(\cdot - \gamma) + \lambda^2 V_{2,\omega,\lambda} \tag{2.6}$$

where

- $v_{\sigma_\gamma} = -\sigma_\gamma \cdot \nabla q$,
- $V_{2,\omega,\lambda}$ is a potential bounded uniformly in λ and ω.

As q is C^2 with compact support, for any $\sigma_0 \in \mathbb{S}^{d-1}$, v_{σ_0} is C^1 with compact support and does not vanish identically. Assumptions (H.0.2) and (H.3) guarantee that the random variables $(r_\gamma(\omega))_{\gamma\in\mathbb{Z}^d}$ are independent and nicely distributed.
Hence, the model (2.6) satisfies the assumptions considered in section 6 of [9] except for the fact that, in the present case, $V_{2,\omega,\lambda}$ depends on λ. This does not matter as it is bounded uniformly in λ. In particular, Lemma 6.1 of [9] from asserts that there exists $\lambda_0 > 0$ such that, for $\lambda \in (0, \lambda_0]$, there exists $C_\lambda > 0$ such that, for all $E \in [E_\lambda, E_\lambda + \lambda/C]$ and $\varepsilon > 0$ such that

$$\mathbb{P}(\mathrm{dist}(\sigma(H^P_{\lambda,\omega,\sigma,n}), E) \leq \varepsilon |\, \sigma) \leq C_\lambda \left[\sup_{\gamma\in\mathbb{Z}^d} \|h'_{\sigma_\gamma}\|_\infty\right] \varepsilon^\nu n^d. \tag{2.7}$$

The explicit form of the the constant appearing on the right side of formula (2.7) is obtained by following the proof of Lemma 6.1 in [9].
The bound (1.4) then guarantees that the sup in (2.7) is essentially bounded as a function of σ. We then complete the proof of Theorem 1.3 by integrating (2.7) with respect to σ and using (2.5). □

3. Proof of Theorem 2.1

We now turn to the proof of Theorem 2.1. The proof follows the spirit of [15, 8].
For $\gamma \in \mathbb{Z}^d$, define $\tilde{\omega} = (\tilde{\omega}_\gamma)_{\gamma\in\mathbb{Z}^d} = (\omega_\gamma - \zeta)_{\gamma\in\mathbb{Z}^d}$. Write

$$V_{\lambda,\omega} = V_{\lambda,\overline{\zeta}} + \lambda\,\delta V_{\lambda,\tilde{\omega}} = V_{\lambda,\overline{\zeta}} + \lambda V_{1,\lambda,\tilde{\omega}} + \lambda^2 V_{2,\lambda,\tilde{\omega}} \tag{3.1}$$

where

$$V_{1,\lambda,\tilde{\omega}} = -\sum_{\gamma\in\mathbb{Z}^d} \nabla q(x-\gamma-\lambda\zeta)\cdot\tilde{\omega}_\gamma. \tag{3.2}$$

We decompose our random Hamiltonian $H_{\lambda,\tilde{\omega}} := H_{\lambda,\omega}$ on the translation-invariant subspaces $\mathcal{E}_{\lambda,\zeta,0}$ and $\mathcal{E}_{\lambda,\zeta,+}$ defined in the section 2.2. Thus, we obtain the random operators

$$H_{\lambda,\tilde{\omega},0} = \Pi_{\lambda,\zeta,0} H_{\lambda,\tilde{\omega}} \Pi_{\lambda,\zeta,0} \quad \text{and} \quad H_{\lambda,\tilde{\omega},+} = \Pi_{\lambda,\zeta,+} H_{\lambda,\tilde{\omega}} \Pi_{\lambda,\zeta,+}.$$

In the orthogonal decomposition of $L^2(\mathbb{R}^d) = \mathcal{E}_{\lambda,\zeta,0} \overset{\perp}{\oplus} \mathcal{E}_{\lambda,\zeta,+}$, $H_{\lambda,\tilde{\omega}}$ is represented by the matrix

$$\begin{pmatrix} H_{\lambda,\tilde{\omega},0} & \lambda\,\Pi_{\lambda,\zeta,0}\,\delta V_{\lambda,\tilde{\omega}}\,\Pi_{\lambda,\zeta,+} \\ \lambda\Pi_{\lambda,\zeta,+}\,\delta V_{\lambda,\tilde{\omega}}\,\Pi_{\lambda,\zeta,0} & H_{\lambda,\tilde{\omega},+}. \end{pmatrix} \tag{3.3}$$

In section 3.1, we give lower and upper bounds on $H_{\lambda,\tilde{\omega},0}$ which we prove in section 3.4. Theorem 2.1 then follows from the fact that the off-diagonal terms in (3.3) are controlled by the diagonal ones; this is explained in section 3.2.

3.1. The operator $H_{\lambda,\tilde{\omega},0}$.

In this section, using the non-degeneracy for the density of states of $H_{\lambda,\overline{\zeta}}$ at $E(\lambda,\zeta)$, we give lower and upper bounds on $H_{\lambda,\tilde{\omega},0}$.
As seen in section 2.2, the operator $H_{\lambda,\tilde{\omega},0}$ is unitarily equivalent to the operator $h_{\lambda,\tilde{\omega}}$ acting on $L^2(\mathbb{T}^*)$ and defined by

$$h_{\lambda,\tilde{\omega}} = h_\lambda + \lambda v_{1,\lambda,\tilde{\omega}} + \lambda^2 v_{2,\lambda,\tilde{\omega}},$$

where

- h_λ is the multiplication by $E_0(\lambda,\zeta,\theta)$,
- the operator $v_{1,\lambda,\tilde{\omega}}$ has the kernel

$$v_{1,\lambda,\tilde{\omega}}(\theta,\theta') = \langle V_{1,\lambda,\tilde{\omega}}\,\varphi_0(\lambda,\zeta,\theta,\cdot), \varphi_0(\lambda,\zeta,\theta',\cdot)\rangle_{L^2(K_0)}, \tag{3.4}$$

- the operator $v_{2,\lambda,\tilde{\omega}}$ has the kernel

$$v_{2,\lambda,\tilde{\omega}}(\theta,\theta') = \langle V_{2,\lambda,\tilde{\omega}}\,\varphi_0(\lambda,\zeta,\theta,\cdot), \varphi_0(\lambda,\zeta,\theta',\cdot)\rangle_{L^2(K_0)}.$$

The potential $V_{1,\lambda,\tilde{\omega}}$ and $V_{2,\lambda,\tilde{\omega}}$ are defined in (3.1) and (3.2). They are bounded uniformly in all parameters. This will be used freely without a special mention.

We now recall a number of facts and definitions taken from [15]. Let $t \in L^2(\mathbb{T}^*, \mathcal{H}_\theta)$. We define the operator $P_t : \ L^2(\mathbb{T}^*) \to L^2(\mathbb{R}^d)$ by

$$\forall u \in L^2(\mathbb{T}^*),\ [P_t(u)](x) = \int_{\mathbb{T}^*} t(\theta, x) u(\theta) d\theta.$$

It satisfies

$$\|P_t\|_{L^2(\mathbb{T}^*) \to L^2(\mathbb{R}^d)} \leq \|t\|_{L^2(\mathbb{T}^*, \mathcal{H}_\theta)}. \tag{3.5}$$

As the Floquet eigenvalue $E_0(\lambda, \zeta, \theta)$ is simple in a neighborhood of 0, the Floquet eigenvector $\varphi_0(\lambda, \zeta, \theta, \cdot)$ is analytic in this neighborhood. Recall that ϖ is defined in (2.2). We define the functions $\varphi_{0,\lambda,\zeta}$, $\tilde{\varphi}_{0,\lambda,\zeta}$ and $\delta\varphi_{0,\lambda,\zeta}$ in $L^2(\mathbb{T}^*, \mathcal{H}_\theta)$ by

$$\varphi_{0,\lambda,\zeta}(\theta, x) = \varphi_0(\lambda, \zeta, \theta; x), \quad \tilde{\varphi}_{0,\lambda,\zeta}(\theta, x) = \varphi_{0,\lambda,\zeta}(0, x) e^{i\theta \cdot x}$$

$$\delta\varphi_{0,\lambda,\zeta}(\theta, x) = \frac{1}{\sqrt{\varpi(\theta)}} (\varphi_0(\lambda, .\zeta, \theta; x) - \tilde{\varphi}_{0,\lambda,\zeta}(\theta, x)).$$

Furthermore, these functions are bounded in $L^2(\mathbb{T}^*, \mathcal{H}_\theta)$ uniformly in ζ and λ small.
Finally, we note that, for $u \in L^2(\mathbb{T}^*)$,

$$P_{\varphi_{0,\lambda,\zeta}}(u) = P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u) + P_{\delta\varphi_{0,\lambda,\zeta}}(\sqrt{\varpi} u). \tag{3.6}$$

Remark 3.1. It is proved in [15] that, there exits $C > 1$ such that, as operators on $L^2(\mathbb{T})$, one has

$$\frac{1}{C} \varpi \ \leq h_\lambda - E(\lambda, \zeta) \leq C\, \varpi.$$

3.1.1. *Lower and upper bounds on $v_{1,\lambda,\tilde{\omega}}$ and $v_{2,\lambda,\tilde{\omega}}$.*

Proposition 3.1. *Recall that $v(\lambda, \zeta)$ is defined in (2.3). There exists $C > 0$ such that, for $u \in L^2(\mathbb{T}^*)$ and $\alpha > 0$, we have*

$$\begin{aligned} &\left| \langle v_{1,\lambda,\tilde{\omega}} u, u \rangle - \sum_{\gamma \in \mathbb{Z}^d} [v(\lambda, \zeta) \cdot \tilde{\omega}_\gamma] \cdot |\hat{u}(\gamma)|^2 \right| \\ &\qquad \leq C \left(\alpha \sum_{\gamma \in \mathbb{Z}^d} \|\tilde{\omega}_\gamma\|^2 \cdot |\hat{u}(\gamma)|^2 + \left(1 + \frac{1}{\alpha}\right) \langle \varpi u, u \rangle \right) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &|\langle v_{2,\lambda,\tilde{\omega}} u, u \rangle| + \|V_{1,\lambda,\tilde{\omega}} P_{\varphi_{0,\lambda,\zeta}}(u)\|^2 + \|V_{2,\lambda,\tilde{\omega}} P_{\varphi_{0,\lambda,\zeta}}(u)\|^2 \\ &\qquad \leq C \left(\sum_{\gamma \in \mathbb{Z}^d} \|\tilde{\omega}_\gamma\|^2 \cdot |\hat{u}(\gamma)|^2 + \langle \varpi u, u \rangle \right). \end{aligned} \tag{3.8}$$

Proposition 3.1 is proved in section 3.4. We now use these results to give lower and upper bounds on $H_{\lambda,\tilde{\omega},0}$.

3.1.2. *Lower and upper bounds for $H_{\lambda,\tilde\omega,0} - E(\lambda,\zeta)$.* We prove

Proposition 3.2. *Under assumptions (H.0) and (H.1), there exists $C_0>0$ such that, for $\alpha>0$, there exists $\lambda_\alpha>0$ and $C_\alpha>0$ such that, for all $\lambda\in[0,\lambda_0]$, on $\mathcal{E}_{\lambda,\zeta,0}$, one has*

$$\frac{1}{C_0}P_{\lambda,\zeta}h^-_{\lambda,\tilde\omega,\zeta}P^*_{\lambda,\zeta} \le \tilde H_{\lambda,\tilde\omega,0} := H_{\lambda,\tilde\omega,0} - E(\lambda,\zeta) \le C_0 P_{\lambda,\zeta}h^+_{\lambda,\tilde\omega,\zeta}P^*_{\lambda,\zeta},$$

where $h^\pm_{\lambda,\tilde\omega,\zeta}$ are the random operators defined in Theorem 2.1.

Proof of Proposition 3.2. For λ small, Proposition 3.1 and Remark 3.1 imply that, there exists $C_0>0$ such that, for $\alpha>0$, there exists $\lambda_\alpha>0$ such that for all $\lambda\in[0,\lambda_\alpha]$,

$$\frac{1}{C_0}\varpi + \lambda\sum_{\gamma\in\mathbb{Z}^d}[v(\lambda,\zeta)\cdot\tilde\omega_\gamma - C_0\alpha\|\tilde\omega_\gamma\|^2]\cdot\Pi_\gamma \le h_{\lambda,\tilde\omega} - E(\lambda,\zeta)$$

and

$$h_{\lambda,\tilde\omega} - E(\lambda,\zeta) \le C_0\,\varpi + \lambda\sum_{\gamma\in\mathbb{Z}^d}[v(\lambda,\zeta)\cdot\tilde\omega_\gamma + C_0\alpha\|\tilde\omega_\gamma\|^2]\cdot\Pi_\gamma.$$

As $H_{\lambda,\tilde\omega,0}$ and $h_{\lambda,\tilde\omega}$ are unitarily equivalent, this completes the proof of Proposition 3.2. □

3.2. **The operator $H_{\lambda,\tilde\omega,+}$.** By the definition of $\Pi_{\lambda,\zeta,+}$, there exists $\eta>0$ such that, for λ sufficiently small,

$$(E(\lambda,\zeta)+\eta)\Pi_{\lambda,\zeta,+} \le \Pi_{\lambda,\zeta,+}H_{\lambda,\overline{\zeta}}\Pi_{\lambda,\zeta,+}.$$

Let $\tilde H_{\lambda,\overline{\zeta},+} = H_{\lambda,\overline{\zeta},+} - E(\lambda,\zeta)$ and $\tilde H_{\lambda,\tilde\omega,+} = H_{\lambda,\tilde\omega,+} - E(\lambda,\zeta)$ (see (3.3)). As $|V_{1,\lambda,\tilde\omega}|$ and $|V_{1,\lambda,\tilde\omega}|$ are bounded, for λ sufficiently small, one has

$$\frac{\eta}{2}\,\Pi_{\lambda,\zeta,+} \le \frac{1}{2}\tilde H_{\lambda,\overline{\zeta},+} \le \tilde H_{\lambda,\tilde\omega,+} \le 2\tilde H_{\lambda,\overline{\zeta},+}. \tag{3.9}$$

3.3. **The proof of Theorem 2.1.** For $\varphi = \varphi_0+\varphi_+ \in \mathcal{E}_{\lambda,\zeta,0}\overset{\perp}{\oplus}\mathcal{E}_{\lambda,\zeta,+}$, by (3.3), one has

$$\begin{aligned}\Big|\langle\tilde H_{\lambda,\tilde\omega}\varphi,\varphi\rangle - \langle\tilde H_{\lambda,\tilde\omega,0}\varphi_0,\varphi_0\rangle - \langle\tilde H_{\lambda,\tilde\omega,+}\varphi_+,\varphi_+\rangle\Big| \\ \le 2\lambda|\langle V_{1,\lambda,\tilde\omega}\varphi_+,\varphi_0\rangle| + 2\lambda^2|\langle V_{2,\lambda,\tilde\omega}\varphi_+,\varphi_0\rangle|.\end{aligned}$$

Using the Cauchy-Schwarz inequality, we get

$$|\langle V_{1,\lambda,\tilde\omega}\varphi_+,\varphi_0\rangle| + |\langle V_{2,\lambda,\tilde\omega}\varphi_+,\varphi_0\rangle| \le 2\|V_{1,\lambda,\tilde\omega}\varphi_0\|^2 + 2\|V_{2,\lambda,\tilde\omega}\varphi_0\|^2 + 4\|\varphi_+\|^2.$$

Then, the decomposition (3.3) and (3.9) give

$$\begin{aligned}\frac{1}{C}\begin{pmatrix}\tilde H_{\lambda,\tilde\omega,0} - C\lambda K_0 & 0\\ 0 & \tilde H_{\lambda,\tilde\omega,+} - C\lambda\Pi_{\lambda,\zeta,+}\end{pmatrix} \le \tilde H_{\lambda,\tilde\omega} \\ = \tilde H_{\lambda,\tilde\omega} \le C\begin{pmatrix}\tilde H_{\lambda,\tilde\omega,0} + C\lambda K_0 & 0\\ 0 & \tilde H_{\lambda,\tilde\omega,+} + C\lambda\Pi_{\lambda,\zeta,+}\end{pmatrix}\end{aligned} \tag{3.10}$$

where

$$K_0 = \Pi_{\lambda,\zeta,0}(V_{1,\lambda,\tilde{\omega}}^2 + V_{2,\lambda,\tilde{\omega}}^2)\Pi_{\lambda,\zeta,0}.$$

The estimate (3.12) of Proposition 3.1 implies that

$$K_0 \leq C P_{\lambda,\zeta} \left(\varpi + \sum_{\gamma\in\mathbb{Z}^d} \|\tilde{\omega}_\gamma\|^2 \Pi_\gamma \right) P_{\lambda,\zeta}^*.$$

On the other hand, (3.9) implies that, for λ sufficiently small

$$\frac{1}{2}\tilde{H}_{\lambda,\tilde{\omega},+} \leq \tilde{H}_{\lambda,\tilde{\omega},+} - C\lambda\Pi_{\lambda,\zeta,+} \leq \tilde{H}_{\lambda,\tilde{\omega},+} + C\lambda\Pi_{\lambda,\zeta,+} \leq 2\tilde{H}_{\lambda,\tilde{\omega},+}.$$

Combining these two estimates with (3.10) and Proposition 3.2, we complete the proof of Theorem 2.1.

3.4. **The proof of Propositions 3.1.** We first prove

Lemma 3.1. *There exists a constant $C > 0$ such that, for all $u \in L^2(\mathbb{T}^d)$ and $\alpha > 0$, one has*

$$\begin{aligned}
&\left| \langle V_{1,\lambda,\tilde{\omega}} P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u), P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\rangle - \sum_{\gamma\in\mathbb{Z}^d} [v(\lambda,\zeta)\cdot\tilde{\omega}_\gamma]\, |\hat{u}(\gamma)|^2 \right| \\
&\qquad \leq C\left(\alpha \sum_{\gamma\in\mathbb{Z}^d} \|\tilde{\omega}_\gamma\|^2 \cdot |\hat{u}(\gamma)|^2 + \left(1 + \frac{4}{\alpha}\right) \langle \varpi u, u\rangle \right),
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&\|V_{1,\lambda,\tilde{\omega}} P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\|^2 + \|V_{2,\lambda,\tilde{\omega}} P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\|^2 \\
&\qquad \leq C\left(\alpha \sum_{\gamma\in\mathbb{Z}^d} \|\tilde{\omega}_\gamma\|^2 \cdot |\hat{u}(\gamma)|^2 + \left(1 + \frac{4}{\alpha}\right) \langle \varpi u, u\rangle \right),
\end{aligned} \tag{3.12}$$

Proof of Lemma 3.1. We compute

$$\begin{aligned}
&\langle V_{1,\lambda,\tilde{\omega}} P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u), P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\rangle \\
&\qquad = -\sum_{\gamma\in\mathbb{Z}^d} \tilde{\omega}_\gamma \cdot \int_{\mathbb{R}^d} \nabla q(x-\lambda\zeta)|\varphi_0(\lambda,\zeta,0;x)|^2 \cdot |\phi_\gamma(u)(x)|^2 dx,
\end{aligned}$$

where $\phi_\gamma(u)(x) = \int_{\mathbb{T}} e^{i\theta\cdot\gamma}\cdot e^{i\theta\cdot x} u(\theta)d\theta$.
Recall that 0 is the unique zero of ϖ on $\mathbb{T}^*$ and it is non-degenerate. Thus, the function $g(\theta,x) = \varpi(\theta)^{-1/2}(e^{i\theta\cdot x} - 1)$ is defined on $\mathbb{T}\times\mathbb{R}^d$ and

$$\sup_{(\theta,x)\in\mathbb{T}^*\times\mathbb{R}} (1+|x|)^{-1}|g(\theta,x)| < +\infty.$$

For $\gamma\in\mathbb{Z}^d$, $u\in L^2(\mathbb{T}^*)$ and $x\in\mathbb{R}^d$, one has

$$\psi_\gamma(u)(x) = \phi_\gamma(u)(x) - \hat{u}(\gamma) = \int_{\mathbb{T}^*} g(\theta,x) e^{i\gamma\cdot\theta}\sqrt{\varpi(\theta)}u(\theta)d\theta.$$

Note that

$$\sum_{\gamma\in\mathbb{Z}^d} |\psi_\gamma(u)(x)|^2 = \int_{\mathbb{T}^*} |g(\theta,x)|^2 |\sqrt{\varpi(\theta)}u(\theta)|^2 d\theta \tag{3.13}$$
$$\leq C(1+|x|)^2\langle \varpi u, u\rangle.$$

Recall that $v(\lambda,\zeta)$ is defined by (2.3). We define

$$v'_{1,\lambda,\tilde{\omega}}[u] = -\sum_{\gamma\in\mathbb{Z}^d} \tilde{\omega}_\gamma \cdot \int_{\mathbb{R}^d} \nabla q(x-\lambda\zeta)|\varphi_0(\lambda,\zeta,0;x)|^2 |\psi_\gamma(u)(x)|^2 dx,$$

$$v''_{1,\lambda,\tilde{\omega}}[u] = \langle V_{1,\lambda,\tilde{\omega}} P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u), P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\rangle - \sum_{\gamma\in\mathbb{Z}^d} v(\lambda,\zeta)\cdot\tilde{\omega}_\gamma|\hat{u}(\gamma)|^2 - v'_{1,\lambda,\tilde{\omega}}[u].$$

As the random variables $(\tilde{\omega}_\gamma)_{\gamma\in\mathbb{Z}^d}$ are bounded, by (3.13), we compute

$$|v'_{1,\lambda,\tilde{\omega}}[u]| \leq C\int_{\mathbb{R}^d} \|\nabla q(x-\lambda\zeta)\||\varphi_0(\lambda,\zeta,0;x)|^2 \sum_{\gamma\in\mathbb{Z}^d}|\psi_\gamma(u)(x)|^2 dx$$
$$\leq C\langle \varpi\, u,u\rangle \int_{\mathbb{R}^d} \|\nabla q(x-\lambda\zeta)\||\varphi_0(\lambda,\zeta,0;x)|^2 (1+|x|)^2 dx$$
$$\leq C\langle \varpi\, u,u\rangle.$$

By the Cauchy-Schwarz inequality, one has

$$|v''_{1,\lambda,\tilde{\omega}}[u]| = 2\left|\mathrm{Re}\left(\sum_{\gamma\in\mathbb{Z}^d}\hat{u}(\gamma)\tilde{\omega}_\gamma\cdot\int_{\mathbb{R}^d}\nabla q(x-\lambda\zeta)|\varphi_0(\lambda,\zeta,0;x)|^2\overline{\psi_\gamma(u)(x)}dx\right)\right|$$
$$\leq \alpha\left[\int_{\mathbb{R}^d}\|\nabla q(x-\lambda\zeta)\|\,|\varphi_0(\lambda,\zeta,0;x)|^2 dx\right]\sum_{\gamma\in\mathbb{Z}^d}\|\tilde{\omega}_\gamma\|^2\,|\hat{u}(\gamma)|^2$$
$$+\frac{4}{\alpha}\sum_{\gamma\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\|\nabla q(x-\lambda\zeta)\|\,|\varphi_0(\lambda,\zeta,0;x)|^2|\psi_\gamma(u)(x)|^2 dx.$$

Using (3.13), we obtain that

$$|v''_{1,\lambda,\tilde{\omega}}[u]| \leq C\left(\alpha\sum_{\gamma\in\mathbb{Z}^d}\|\tilde{\omega}_\gamma\|^2\cdot|\hat{u}(\gamma)|^2 + \frac{4}{\alpha}\langle\varpi u,u\rangle\right).$$

Finally, adding this to the estimate for $|v'_{1,\lambda,\tilde{\omega}}[u]|$, we get

$$\left|\langle V_{1,\lambda,\tilde{\omega}} P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u), P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\rangle - \sum_{\gamma\in\mathbb{Z}^d} v(\lambda,\zeta)\,\cdot\tilde{\omega}_\gamma|\hat{u}(\gamma)|^2\right|$$
$$\leq C\left(\alpha\sum_{\gamma\in\mathbb{Z}^d}\|\tilde{\omega}_\gamma\|^2\cdot|\hat{u}(\gamma)|^2 + \left(1+\frac{4}{\alpha}\right)\langle\varpi u,u\rangle\right).$$

This completes the proof of (3.11).
The two terms in the left hand side of (3.12) are dealt with in the same way; so, we only give the details for $\|V_{1,\lambda,\tilde{\omega}} P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\|^2$. We compute

$$\begin{aligned}
\|V_{1,\lambda,\tilde{\omega}} P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\|^2 &= \int_{\mathbb{R}^d} |V_{1,\lambda,\tilde{\omega}}(x)\, \varphi_0(\lambda,\zeta,0;x)\, \phi(u)(x)|^2\, dx \\
&= \int_{\mathbb{R}^d} \left| \sum_{\gamma\in\mathbb{Z}^d} \nabla q(x-\lambda\zeta-\gamma)\cdot\tilde{\omega}_\gamma \right|^2 |\varphi_0(\lambda,\zeta,0;x)\, \phi(u)(x)|^2 dx \\
&\le C \sum_{\gamma\in\mathbb{Z}^d} \|\tilde{\omega}_\gamma\|^2 \int_{\mathbb{R}^d} \|\nabla q(x-\lambda\zeta)\|^2\, |\varphi_0(\lambda,\zeta,0;x)\, \phi_\gamma(u)(x)|^2 dx
\end{aligned}$$

where, in the last step, as q is compactly supported, the number of non vanishing terms of the sum inside the integral is bounded uniformly.
Now, by the definition of ϕ_γ and ψ_γ, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\nabla q(x-\lambda\zeta)|^2 |\varphi_0(\lambda,\zeta,0;x)|^2 |\phi_\gamma(u)(x)|^2 dx \\
&\qquad \le 2 \int_{\mathbb{R}^d} |\nabla q(x-\lambda\zeta)|^2 |\varphi_0(\lambda,\zeta,0;x)|^2 \left(|\hat{u}(\gamma)|^2 + |\psi_\gamma(u)(x)|^2\right) dx \\
&\qquad \le C|\hat{u}(\gamma)|^2 + C \int_{\mathbb{R}^d} |\nabla q(x-\lambda\zeta)|^2 |\varphi_0(\lambda,\zeta,0;x)|^2 |\psi_\gamma(u)(x)|^2 dx.
\end{aligned}$$

We plug this into the estimate for $\|V_{1,\lambda,\tilde{\omega}} P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\|^2$ and (3.13) yields

$$\begin{aligned}
&\|V_{1,\lambda,\tilde{\omega}} P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\|^2 \\
&\qquad \le C \int_{\mathbb{R}^d} |\nabla q(x-\lambda\zeta)|^2 |\varphi_0(\lambda,\zeta,0;x)|^2 \cdot \sum_{\gamma\in\mathbb{Z}^d} |\phi_\gamma(u)(x)|^2 dx \\
&\qquad\qquad\qquad\qquad + C \sum_{\gamma\in\mathbb{Z}^d} \|\tilde{\omega}_\gamma\|^2 |\hat{u}(\gamma)|^2 \\
&\qquad \le C\|\tilde{\omega}_\gamma\|^2 |\hat{u}(\gamma)|^2 + C\langle \varpi u, u\rangle.
\end{aligned}$$

The computation for $\|V_{2,\lambda,\tilde{\omega}} P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\|^2$ is the same as

$$V_{2,\lambda,\tilde{\omega}} = \sum_{\gamma\in\mathbb{Z}^d} q(x-\gamma,\zeta,\tilde{\omega}_\gamma)$$

where, denoting the Hessian of q at x by Q, we have

$$q(x,\zeta,\tilde{\omega}_\gamma) = \int_0^1 \langle Q(x-\lambda(\zeta+t\tilde{\omega}_\gamma))\tilde{\omega}_\gamma, \tilde{\omega}_\gamma\rangle (1-t)dt.$$

So (3.12) is proved and the proof of Lemma 3.1 is complete. $\square$

Proof of Proposition 3.1. Using (3.4) and (3.6), we write

$$\begin{aligned}\langle v_{1,\lambda,\tilde{\omega},\zeta}u,u\rangle = {} & \langle V_{1,\lambda,\tilde{\omega}}P_{\delta\varphi_{0,\lambda,\zeta}}(\sqrt{\varpi}u), P_{\delta\varphi_{0,\lambda,\zeta}}(\sqrt{\varpi}u)\rangle \\ & + \langle V_{1,\lambda,\tilde{\omega}}P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u), P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\rangle \\ & + 2\mathrm{Re}(\langle V_{1,\lambda,\tilde{\omega}}P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u), P_{\delta\varphi_{0,\lambda,\zeta}}(\sqrt{\varpi}u)\rangle).\end{aligned}$$

Hence, as $V_{1,\lambda,\tilde{\omega}}$ is bounded,

$$\begin{aligned} & \left|\langle v_{1,\lambda,\tilde{\omega}}u,u\rangle - \sum_{\gamma\in\mathbb{Z}^d} v(\lambda,\zeta)\cdot\tilde{\omega}_\gamma|\hat{u}(\gamma)|^2\right| \\ & \leq \left|\langle V_{1,\lambda,\tilde{\omega}}P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u), P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\rangle - \sum_{\gamma\in\mathbb{Z}^d} v(\lambda,\zeta)\cdot\tilde{\omega}_\gamma|\hat{u}(\gamma)|^2\right| \\ & \quad + 2\left|\langle V_{1,\lambda,\tilde{\omega}}P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u), P_{\delta\varphi_{0,\lambda,\zeta}}(\sqrt{\varpi}u)\rangle\right| + \left\|P_{\delta\varphi_{0,\lambda,\zeta}}(\sqrt{\varpi}u)\right\|^2.\end{aligned} \tag{3.14}$$

The Cauchy-Schwarz inequality then yields

$$\begin{aligned} & |\langle V_{1,\lambda,\tilde{\omega}}P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u), P_{\delta\varphi_{0,\lambda,\zeta}}(\sqrt{\varpi}u)\rangle| \\ & \qquad \leq \alpha\|V_{1,\lambda,\tilde{\omega}}P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\|^2 + \frac{4}{\alpha}\|P_{\delta\varphi_{0,\lambda,\zeta}}(\sqrt{\varpi}u)\|^2.\end{aligned}$$

Combining this with (3.14), (3.12) and (3.11), we obtain

$$\begin{aligned} & \left|\langle v_{1,\lambda,\tilde{\omega}}u,u\rangle - \sum_{\gamma\in\mathbb{Z}^d}[v(\lambda,\zeta)\cdot\tilde{\omega}_\gamma]\cdot|\hat{u}(\gamma)|^2\right| \\ & \qquad \leq C\left(\alpha\sum_{\gamma\in\mathbb{Z}^d}\|\tilde{\omega}_\gamma\|^2\cdot|\hat{u}(\gamma)|^2 + \left(1+\frac{1}{\alpha}\right)\langle\varpi u,u\rangle\right).\end{aligned}$$

This completes the proof of (3.7).
Using (3.6) and the expansion done above for $\langle v_{1,\lambda,\tilde{\omega}}u,u\rangle$, we compute

$$|\langle v_{2,\lambda,\tilde{\omega}}u,u\rangle| \leq 2\|V_{2,\lambda,\tilde{\omega}}P_{\tilde{\varphi}_{0,\lambda,\zeta}}(u)\|^2 + 2\|P_{\delta\varphi_{0,\lambda,\zeta}}(\sqrt{\varpi}u)\|^2.$$

Combining this with (3.12), we get

$$|\langle v_{2,\lambda,\tilde{\omega}}u,u\rangle| \leq C\left(\sum_{\gamma\in\mathbb{Z}^d}\|\tilde{\omega}_\gamma\|^2\cdot|u(\gamma)|^2 + \langle\varpi u,u\rangle\right).$$

The estimates for $\|V_{1,\lambda,\tilde{\omega}}P_{\varphi_{0,\lambda,\zeta}}(u)\|^2$ and $\|V_{2,\lambda,\tilde{\omega}}P_{\varphi_{0,\lambda,\zeta}}(u)\|^2$ are obtained in the same way. This completes the proof of (3.8), hence, of Proposition 3.1. □

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(Fatma Ghribi) Département de Mathématiques, Faculté des Sciences de Monastir, Avenue de l'Environnement 5019 Monastir, Tunisie
E-mail address: Fatma.Ghribi@fsm.rnu.tn

(Frédéric Klopp) LAGA, Institut Galilée, U.R.A 7539 C.N.R.S, Université de Paris-Nord, Avenue J.-B. Clément, F-93430 Villetaneuse, France, et, Institut Universitaire de France
E-mail address: klopp@math.univ-paris13.fr